On the length of attractors in boolean networks with an interaction graph by layers.

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Abstract

We consider a boolean network whose interaction graph has no circuit of length ≥ 2. Under this hypothesis, we establish an upper bound on the length of the attractors of the network which only depends on its interaction graph.

1 Introduction

We consider a boolean network $F : \{0,1\}^n \to \{0,1\}^n$ and its interaction graph $G(F)$. The vertices correspond to the components of the network, and there is a positive (resp. negative) edge from j to i if the component j has a positive (resp. negative) effect on the component i. Then, under the assumption that $G(F)$ has no circuit of length > 1 (directed graphs without circuit of length ≥ 2 are called graph by layers in [1]), we establish an upper bound on the length of the attractor of the network which only depends on $G(F)$. This result is related to a recent work of Goles and Salinas [1].

2 Definitions

Let n be a positive integer, and let F be a map from $\{0,1\}^n$ to itself:

$$x = (x_1, \dots, x_n) \in \{0,1\}^n \;\mapsto\; F(x) = (f_1(x), \dots, f_n(x)) \in \{0,1\}^n.$$

As usual, we see F has as a synchronous boolean network with n components: when the network is in state x at time t, it is in state $F(x)$ at time $t+1$.

A *path* of F of length $r \geq 1$, is a sequence $(x^0, x^1, \dots, x^r)$ of points of $\{0,1\}^n$ such that $F(x^k) = x^{k+1}$ for all $0 \leq k < r$. A *cycle* of F of length $r \geq 1$ is a path $(x^0, x^1, \dots, x^r)$ such that $x^0 = x^r$ and such that the points $x^0, \dots, x^{k-1}$ are pairwise distinct. The cycles of F correspond to the attractors of the network.

We set $\bar{0} = 1$ and $\bar{1} = 0$. Then, for all $x \in \{0,1\}$, we denote by $\bar{x}^i$ the points y of $\{0,1\}^n$ defined by $y_i = \bar{x}_i$ and $y_j = x_j$ for all $j \neq i$. For all $x \in \{0,1\}^n$, we set:

$$f_{ij}(x) = \frac{f_i(\bar{x}^j) - f_i(x)}{\bar{x}_j - x_j} \qquad (i, j = 1, \dots, n).$$

f_{ij} may be see has the partial derivative of f_i with respect to the variable x_j.

We are now in position to define the interaction graph of the network: the *interaction graph of F*, denoted $G(F)$, is the graph whose set of vertices is $\{1,\dots,n\}$ and which contains an edge from j to i of sign $s\in\{-1,1\}$ if there exists $x\in\{0,1\}^n$ such that $s=f_{ij}(x)$. So each edge of $G(F)$ is directed and labelled with a sign, and $G(F)$ can contains both a positive and a negative edge from one vertex to another. Note that there exists an edge from j to i in $G(F)$ if and only if f_i depends on x_j.

Let i,j be two vertices of $G(F)$. We say that i is a *successor* (resp. *predecessor*) of j if $G(F)$ has an edge from j to i (resp. from i to j). We say that i is a *strict successor* (resp. *strict predecessor*) of j if i is a successor (resp. predecessor) of j and $i\neq j$. A path of $G(F)$ of length $r\geq 0$ is a sequence $P=(i_0,\dots,i_r)$ of vertices of $G(F)$ such that i_{k+1} is a successor of i_k for all $0\leq k<r$. We say that P is a path from i_0 to i_r, and that P is *elementary* if the vertices $i_0,\dots,i_r$ are pairwise distinct. A *circuit* of $G(F)$ of length $r\geq 1$ is a path $(i_0,\dots,i_r)$ such that $i_0=i_r$ and such that the vertices $i_0,\dots,i_{r-1}$ are pairwise distinct. A positive (resp. negative) edge from a vertex i to itself is called a *positive (resp. negative) loop* on i.

Definition 1 *Let P be an elementary path of $G(F)$. We denote by $\tau_{G(F)}(P)$ the number of vertices i in P satisfying at least one of the two following properties:*

1. *i is the first vertex of P with a negative loop;*
2. *i has both a positive and a negative loop.*

We set $\tau(G(F))=\max\{\tau_{G(F)}(P),\ P$ is an elementary path of $G(F)\}$.

See Figure 1 for an illustration of this definition. Note that $\tau(G(F))\geq 1$ if and only if $G(F)$ has a negative loop, and that $\tau(G(F))\leq 1$ if there is no vertex with both a positive and a negative loop.

3 Result

Goles and Salinas [1] proved the following theorem:

Theorem 1 *Let $F:\{0,1\}^n\to\{0,1\}^n$ be such that $G(F)$ has no circuit of length ≥ 2. If F has a cycle, then the length of this cycle if a power of two, and it is 1 if $G(F)$ has no negative loop.*

The aim of this note is to prove the following extension:

Theorem 2 *Let $F:\{0,1\}^n\to\{0,1\}^n$ be such that $G(F)$ has no circuit of length ≥ 2. If F has a cycle, then the length of this cycle is a power of two less than or equal to $2^{\tau(G(F))}$.*

The proof needs few additional definitions. Let F and $\tilde{F}$ be two maps from $\{0,1\}^n$ to itself. We say that $G(\tilde{F})$ is a *subgraph* (resp. a *strict subgraph*) of $G(F)$ if the set of edges of $G(\tilde{F})$ is a subset (resp. a strict subset) of the set of edges of $G(F)$. We say that F is *r-minimal* if F has a cycle of length r and if there is no map $\tilde{F}$ with a cycle of length r such that $G(\tilde{F})$ is a strict subgraph of $G(F)$. Note that if F has a cycle of length r, there always exists a r-minimal map $\tilde{F}$ such that $G(\tilde{F})$ is a subgraph of $G(F)$.

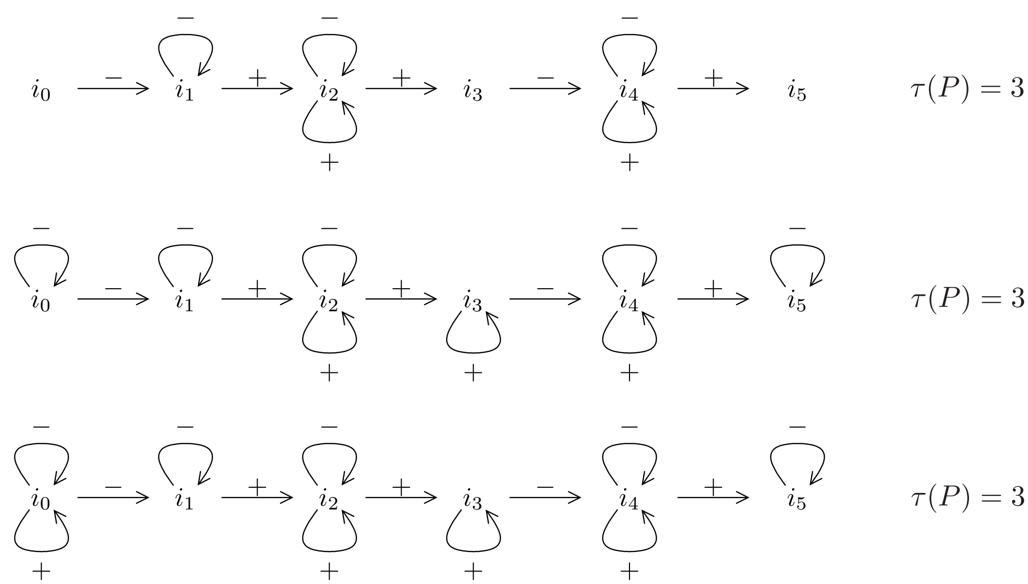


Figure 1: Illustration of Definition 1.

Lemma 1 *Let $F : \{0,1\}^n \to \{0,1\}^n$ be r-minimal with $r \geq 2$, and assume that $G(F)$ has no circuit of length ≥ 2. There exists a map $\tilde{F} : \{0,1\}^n \to \{0,1\}^n$ with a cycle of length $r/2$ such that $G(\tilde{F})$ is a subgraph of $G(F)$ and such that $\tau(G(\tilde{F})) < \tau(G(F))$.*

Proof – Let $\sigma = (x^0, \dots, x^r)$ be a cycle of F of length r. To simplify notations, we set $x^{k+r} = x^k$ for all positive integer k. Since σ is of length ≥ 2, F is not constant. Thus, there exists a vertex j in $G(F)$ with a predecessor. Let P be an elementary path of $G(F)$ of maximal length starting from j, and let i be the last vertex of this path. Then:

The vertex i has a predecessor and no strict successor in $G(F)$.

The fact that i has a predecessor is obvious if $i \neq j$, and true by hypothesis if $i = j$; i has no strict successor since if not, the path P being elementary and of maximal length, $G(F)$ would have a circuit of length ≥ 2.

Let $\tilde{F} : \{0,1\}^n \to \{0,1\}^n$ be defined by:

$$\tilde{f}_i = \text{cst} = 0, \qquad \tilde{f}_j = f_j \quad \text{for all} \quad j \neq i.$$

It is easy to see that $G(\tilde{F})$ is the subgraph of $G(F)$ that we obtain by removing all the edges whose end vertex is i. Since i has a predecessor in $G(F)$, we deduce that:

$G(\tilde{F})$ is a strict *subgraph of $G(F)$.*

In the following, we prove that $\tilde{F}$ has a cycle of length $r/2$ and that $\tau(G(\tilde{F})) < \tau(G(F))$.

For all integer k, let $\tilde{x}^k$ be the point of $\{0,1\}^n$ defined by:

$$\tilde{x}_i^k = 0, \qquad \tilde{x}_j^k = x_j^k \quad \text{for all} \quad j \neq i.$$

Since $\tilde{f}_j = f_j$ does not depend on x_i for all $j \neq i$ (vertex i has no strict successor in $G(F)$), and since $\tilde{f}_i = \text{cst}$ does not depend on x_i, we have $\tilde{F}(\tilde{x}^k) = \tilde{F}(x^k)$ and we deduce that:

$$\tilde{F}(\tilde{x}^k) = \tilde{F}(x^k) = (0, f_2(x^k), \dots, f_n(x^k)) = (0, x_2^{k+1}, \dots, x_n^{k+1}) = \tilde{x}^{k+1}.$$

In other words, $(\tilde{x}^0, \dots, \tilde{x}^r)$ is a path of $\tilde{F}$. Since $\tilde{x}^0 = \tilde{x}^r$, we deduce that $\tilde{F}$ has a cycle $(\tilde{x}^0, \dots, \tilde{x}^p)$ of length $p \leq r$. Then, for all integer k, we have:

$$\tilde{x}^{k+p} = \tilde{x}^k. \tag{1}$$

Since $G(\tilde{F})$ is a strict subgraph of $G(F)$, and since F is r-minimal, we have $p < r$. Consequently, for all integer k:

$$x^{k+p} \neq x^k.$$

From this and (1), we deduce that, for all integer k:

$$x^{k+p} = \overline{x^k}^i. \tag{2}$$

Consequently,

$$x^{k+2p} = \overline{x^{k+p}}^i = \overline{\overline{x^k}^i}^i = x^k$$

and we deduce that $2p = r$: $\tilde{F}$ has indeed a cycle of length $r/2$.

Let j be any vertex of $G(F)$ with a predecessor and without strict successor. With similar argument, we can show that $x^{k+r/2} = \overline{x^k}^j$. Then $\overline{x^k}^j = \overline{x^k}^i$ so that $i = j$. Consequently:

$$\begin{array}{c}\textit{The vertex } i \textit{ is the } \text{unique} \textit{ vertex of } G(F) \\ \textit{with a predecessor and without strict successor.}\end{array} \tag{3}$$

We deduce that:

$$\textit{If a vertex } j \textit{ has a predecessor in } G(F)\textit{, then } G(F) \textit{ has a path from } j \textit{ to } i. \tag{4}$$

Indeed, let j be a vertex with a predecessor, and let P an elementary path of $G(F)$ of maximal length starting from j. As argued above, the last vertex of P has a predecessor and no strict successor. We then deduce from (3) that the last vertex of P is i.

Now, we prove that:

$$\textit{The vertex } i \textit{ has a negative loop in } G(F). \tag{5}$$

Since $x^p = \overline{x^0}^i$, we have $x_i^0 \neq x_i^p$, and we deduce that there exists $0 \leq k < p$ such that:

$$x_i^k \neq x_i^{k+1}.$$

Then:

$$f_i(x^k) = x_i^{k+1} = \overline{x_i^k} = x_i^{k+p}.$$

Moreover, we have

$$x_i^{k+1+p} \neq x_i^{k+1}$$

so

$$f_i(x^{k+p}) = x_i^{k+p+1} = \overline{x_i^{k+1}} = x_i^k$$

and using (2) we deduce that:

$$f_{ii}(x^k) = \frac{f_i(x^{k+p}) - f_i(x^k)}{x_i^{k+p} - x_i^k} = \frac{x_i^k - x_i^{k+p}}{x_i^{k+p} - x_i^k} = -1.$$

In addition:

$$\textit{If } i \textit{ has a strict predecessor in } G(F)\textit{, then } i \textit{ has a positive loop in } G(F). \tag{6}$$

Suppose that i has a strict predecessor, and suppose that $x_i^k \neq x_i^{k+1}$ for all k. Consider the map $\bar{F} : \{0,1\}^n \to \{0,1\}^n$ defined by $\bar{f}_i(x) = \bar{x}_i$ and $\bar{f}_j = f_j$ for $j \neq i$. Clearly, σ is a cycle of $\bar{F}$, and $G(\bar{F})$ is the subgraph of $G(F)$ that we obtain by removing the edges whose end vertex is i, expect the negative loop on i (whose existence is proved). Since i has a strict predecessor in $G(F)$, we deduce that $G(\bar{F})$ is a strict subgraph of $G(F)$, and this is not possible since F is r-minimal. Thus there exists k such that

$$x_i^k = x_i^{k+1} = f_i(x^k).$$

Then

$$x_i^{k+p} \neq x_i^k = x_i^{k+1} \qquad \text{and} \qquad x_i^{k+p+1} \neq x_i^{k+1}$$

so

$$x_i^{k+p} = x_i^{k+p+1} = f_i(x^{k+p})$$

and using (2) we deduce that:

$$f_{ii}(x^k) = \frac{f_i(x^{k+p}) - f_i(x^k)}{x_i^{k+p} - x_i^k} = \frac{x_i^{k+p} - x_i^k}{x_i^{k+p} - x_i^k} = 1.$$

We are now in position to prove that $\tau(G(\tilde{F})) < \tau(G(F))$. Since i has a negative loop in $G(F)$, we have $\tau(G(F)) > 0$. So suppose that $\tau(G(\tilde{F})) > 0$, and let P be an elementary path of $G(\tilde{F})$ such that

$$\tau_{G(\tilde{F})}(P) = \tau(G(\tilde{F})).$$

Since $G(\tilde{F})$ is a subgraph of $G(F)$, P is an elementary path of $G(F)$ and

$$\tau_{G(\tilde{F})}(P) \leq \tau_{G(F)}(P).$$

Let j be the first vertex of P with a negative loop in $G(\tilde{F})$ (j exists since $\tau(G(\tilde{F})) > 0$), and let k be the last vertex of P. Then k has a predecessor in $G(\tilde{F})$ (this is obvious if $k \neq j$ and also true if $k = j$ since j has a negative loop) and thus $k \neq i$ (since i has no predecessor in $G(\tilde{F})$). So k has a predecessor in $G(F)$ and following (4), there exists an elementary path P' from k to i in $G(F)$. Since $G(F)$ has no circuit of length ≥ 2, the concatenation Q of P and P' is an elementary path of $G(F)$, and since $k \neq i$, i has a strict predecessor in $G(F)$. We then deduce from (5) and (6) that i has both a positive and a negative loop in $G(F)$. It is then clear that

$$\tau(G(\tilde{F})) \leq \tau_{G(F)}(P) < \tau_{G(F)}(Q) \leq \tau(G(F))$$

□

Proof of Theorem 2 – Let $F : \{0,1\}^n \to \{0,1\}^n$ be such that $G(F)$ has no circuit of length ≥ 2 and suppose that F has a cycle of length r. We want to prove that r is a power of two less than or equal to $2^{\tau(G(F))}$. We proceed by induction on r. The base case $r = 1$ is obvious. So suppose that $r > 1$. The induction hypothesis is:

> Let $\tilde{F} : \{0,1\}^n \to \{0,1\}^n$ be such that $G(\tilde{F})$ has no circuit of length ≥ 2.
> If $\tilde{F}$ has a cycle of length $l < r$, then l is a power of two $\leq 2^{\tau(G(\tilde{F}))}$.

Consider a r-minimal map $\bar{F} : \{0,1\}^n \to \{0,1\}^n$ such that $G(\bar{F})$ is a subgraph of $G(F)$. Then $G(\bar{F})$ has no circuit of length ≥ 2, and following Lemma 1, there exists a map $\tilde{F}$ with a cycle of length $r/2$ such that $G(\tilde{F})$ is a subgraph of $G(F)$ and such that $\tau(G(\tilde{F})) < \tau(G(F))$. Since $G(\tilde{F})$ is a subgraph $G(\bar{F})$, $G(\tilde{F})$ has no circuit of length ≥ 2. So, by induction hypothesis, $r/2$ is a power of two $\leq 2^{\tau(G(\tilde{F}))}$. So r is a power of two, and since $\tau(G(\tilde{F})) < \tau(G(\bar{F}))$ we have $r \leq 2^{\tau(G(\bar{F}))}$. Since $G(\bar{F})$ is a subgraph of $G(F)$, we have $\tau(G(\bar{F})) \leq \tau(G(F))$ and we deduce that $r \leq 2^{\tau(G(F))}$. □

Let us say that $G(F)$ has an *ambiguous loop*, if $G(F)$ has a vertex with both a positive and a negative loop.

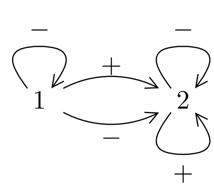
Corollary 1 *Let $F : \{0,1\}^n \to \{0,1\}^n$ be such that $G(F)$ has no circuit of length ≥ 2. If $G(F)$ has no ambiguous loop, then F has no cycle of length ≥ 3.*

Proof – Under the conditions of the statement, it is clear that $\tau(G(F)) \leq 1$. So following Theorem 2, all the cycles of F are of length ≤ 2. □

Remark 1 *In* [2, page 292], *Robert proposes to study the following assertion: If each vertex of $G(F)$ has a loop, and if $G(F)$ has no circuit of length ≥ 2, then F has no cycle of length ≥ 3. This assertion is false as showed by the following example. Let $F : \{0,1\}^2 \to \{0,1\}^2$ be defined by:*

$$F(0,0) = (1,0), \quad F(1,0) = (0,1), \quad F(0,1) = (1,1), \quad F(1,1) = (0,0).$$

F has clearly a cycle of length 4, but each vertex of $G(F)$ has a loop, and $G(F)$ has no circuit of length ≥ 2. The interaction graph $G(F)$ is indeed the following:



According to the previous corrolary, the following assertion, near that the one that Robert proposes to study, is true: If each vertex of $G(F)$ has a loop, and if $G(F)$ has no circuit of length ≥ 2 and no ambiguous loop, then F has no cycle of length ≥ 3.

References

[1] E. Goles, L. Salinas, Comparison between parallel and serial dynamics of Boolean networks, Theoretical Computer Science 396:247-253, 2008.

[2] F. Robert, Les systèmes dynamiques discrets, in: Mathématiques et Applications, Vol. 19, Springer-Verlag, Berlin-Heidelber-New York, 1995.